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A Vortex Model for Transport in the Polar Stratosphere

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I. J. Eberstein
Laboratory for Atmospheric Sciences
Goddard Space Flight Center
Greenbelt, Maryland 20771

F. Y. Yap
Computer Sciences Corporation
Silver Spring, Maryland 20901

V. Veirs
Colorado College
Colorado Springs, Colorado

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GODDARD SPACE FLIGHT CENTER
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ABSTRACT

A semi-empirical model based on a Gaussian vorticity distribution has been developed for determining eddy diffusivity and wind transport distributions in the polar stratosphere. The model uses as input data pressure surface heights measured at periods of the year when the stratospheric polar vortex exhibits nearly circular patterns around the pole. The components of the polar wind velocities that result from a Prandtl eddy viscosity distribution are found to be in general agreement with those obtained by other investigators.

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ILLUSTRATIONS

A VORTEX MODEL FOR TRANSPORT IN THE POLAR STRATOSPHERE

A. INTRODUCTION

A vortex based model has been developed to calculate the eddy diffusivity distribution and the components of the wind velocities in the polar stratosphere and mesosphere in the altitude range 15–60 km. The model is semi-empirical in its application, and requires input data of measured pressure surface heights between 100 mb and 0.4 mb during the late fall–early winter months when the pressure contours exhibit well developed and relatively undistorted circular patterns centered at or near the north pole. In addition to determining the wind transport, the model is also useful in investigating the relationship between eddy viscosity and mean flow, and the effect of these parameters on polar stratospheric ozone.

This paper describes the mathematical development of the polar vortex model, and the numerical computer techniques that are employed in performing the calculations. Results generated by the model are presented and discussed.

B. THE EQUATIONS OF MOTION

The geometric configuration and coordinate system that are used to develop the polar vortex wind model are shown in Figures 1 and 2. The coordinate system is fixed to a point of observation O at latitude λ on the surface of the Earth which is rotating with angular velocity $\vec{\omega}$. In this rotating coordinate system the velocity vector $\vec{V}$ of a particle may be written in cylindrical coordinates (r, ϕ, z) as

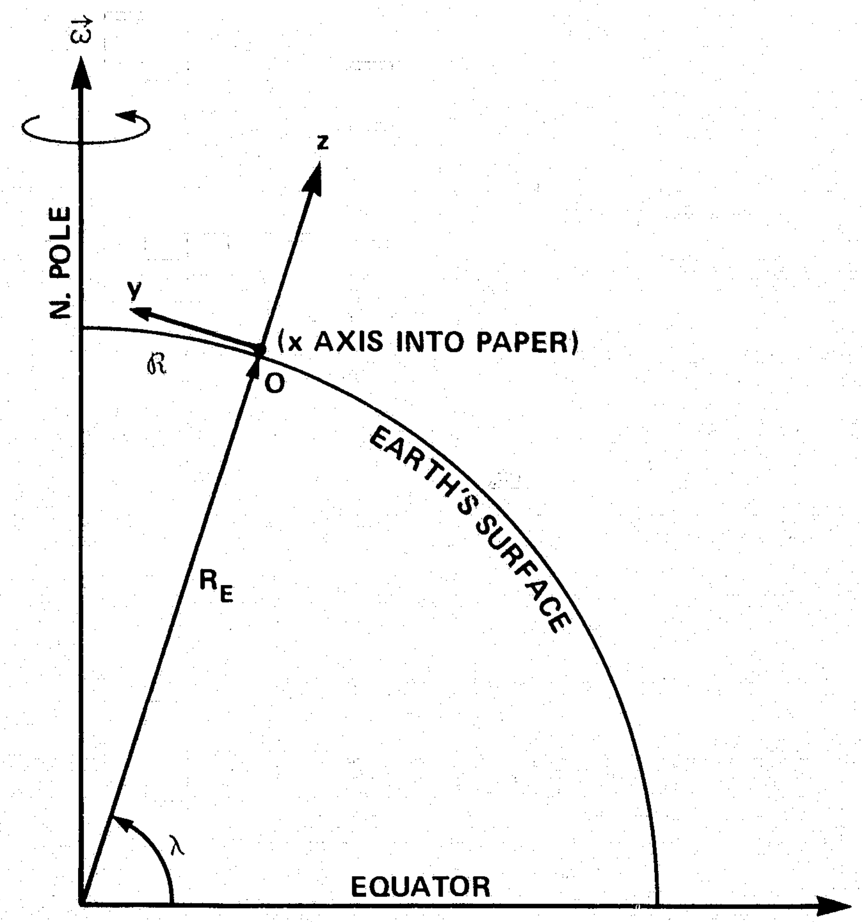

Figure 1. Geometric Configuration of a Rotating Coordinate System Fixed to the Earth's Surface

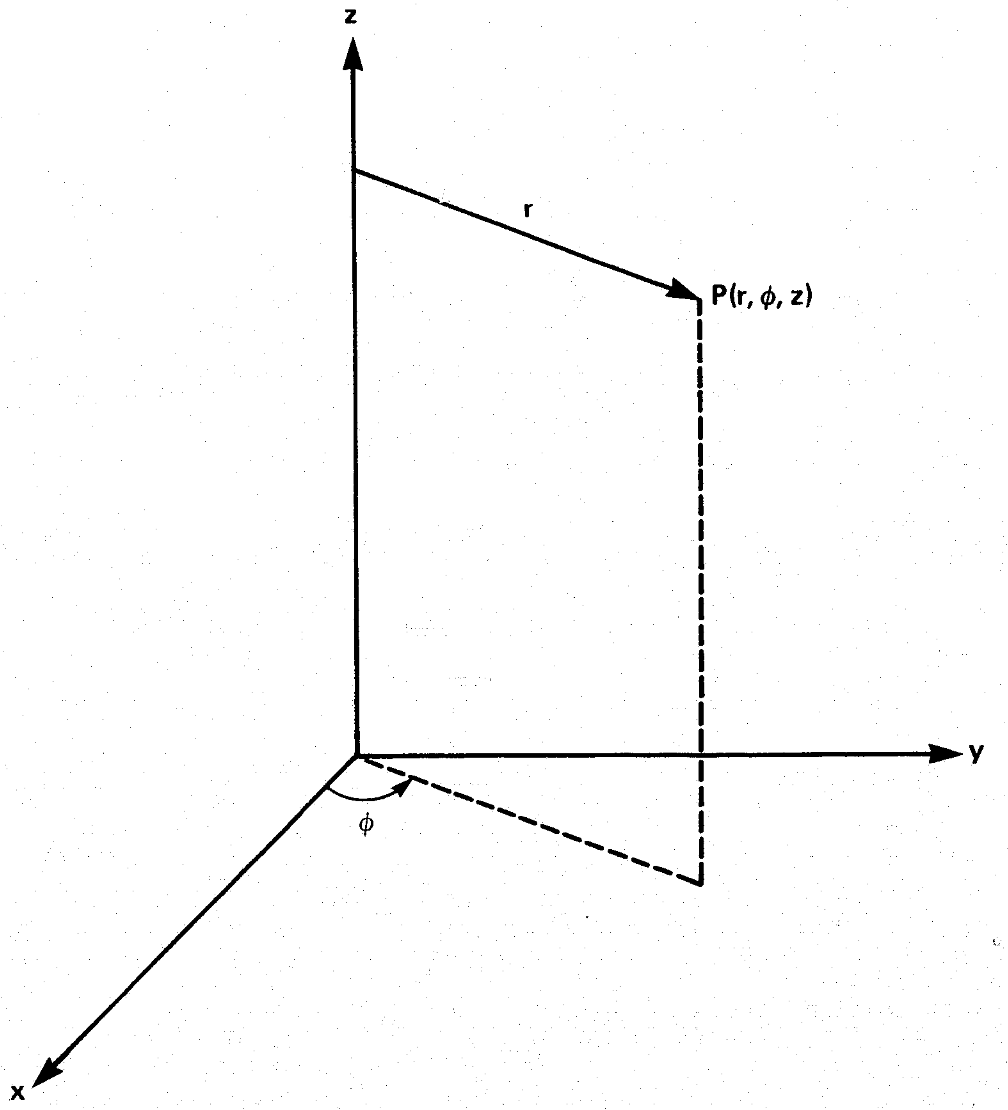

Figure 2. Cylindrical Coordinate System Used in the Development of the Polar Vortex Wind Model

$$\vec{V} = v\hat{r} + u\hat{\phi} + w\hat{z}, \tag{1}$$

where

$\hat{r}$, $\hat{\phi}$ and $\hat{z}$ are unit vectors in the r, ϕ and z directions, respectively,

$u \equiv$ the zonal component of velocity,

$v \equiv$ the meridional component of velocity,

$w \equiv$ the vertical component of velocity.

We note that at large latitudes λ when O is close to the north pole, the curved distance $\mathcal{R}$ from the pole to O closely coincides with the r axis of the rotating cylindrical coordinate system.

The general equation of motion for an element of unit mass, with respect to the rotating coordinate system is (Craig, 1960):

$$\vec{a} \equiv \frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}P - 2\vec{\omega} \times \vec{V} + \vec{g} + \vec{f}, \tag{2}$$

where

$\vec{a}$ is the acceleration of the unit mass of air,

$\frac{1}{\rho}\vec{\nabla}P$ is the force due to the pressure gradient,

$-2(\vec{\omega} \times \vec{V})$ is the Coriolis force due to the Earth's rotation,

$\vec{g}$ is the force due to gravity, which acts only in z direction,

$\vec{f}$ is the frictional force,

ρ is the air density.

We may express certain terms in the above equation in cylindrical coordinates as follows:

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

$$\vec{\nabla}P = \frac{\partial P}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial P}{\partial \phi}\hat{\phi} + \frac{\partial P}{\partial z}\hat{z}$$

$$\vec{\omega} \times \vec{V} = - u\omega\sin\lambda\hat{r} + (v\omega\sin\lambda + w\omega\cos\lambda)\hat{\phi} - u\omega\cos\lambda\hat{z}.$$

Therefore, Equation (2), in component form, becomes:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{1}{\rho}\frac{\partial P}{\partial r} + 2u\omega\sin\lambda + f_r \tag{3a}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\frac{1}{\rho r}\frac{\partial P}{\partial \phi} - 2(v\omega\sin\lambda + w\omega\cos\lambda) + f_\phi \tag{3b}$$

$$\ddot{z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 2u\omega\cos\lambda + g + f_z. \tag{3c}$$

C. THE ZONAL WIND SPEED

The vortex model assumes a vorticity ξ which has a Gaussian distribution with axial symmetry about the center of the coordinate system near the north pole:

$$\zeta = \xi_o\, e^{-\frac{r^2}{2\sigma^2}} \tag{4}$$

where r is the radial distance from the center and σ is the width of the distribution. From the definition of vorticity $\zeta = \overset{o}{\vec{\nabla}} \times \vec{V}$, Stokes' Theorem may be applied around a cylindrical contour of constant zonal speed:

$$\iint_\zeta \hat{n} \cdot \overset{o}{\vec{\nabla}} \times \vec{V} ds = \int_o^R \xi_o\, e^{-\frac{r^2}{2\sigma^2}}\, 2\pi r dr = \oint \vec{V} \cdot d\vec{l} = 2\pi R u \tag{5}$$

where u is the zonal component of the wind velocity and R is the radial distance from the center at the point where the zonal speed is to be calculated.

Integrating Equation (5) we obtain

$$\frac{2\pi\xi_o}{2\left(\frac{1}{2\sigma^2}\right)}\left[1 - e^{-\frac{R^2}{2}}\right] = 2\pi R u\,. \tag{6}$$

We may define the vortex strength K and vortex radius A as

$$K \equiv \frac{1}{2}\,\xi_o\,\sigma^2$$

$$A \equiv \sigma\,. \tag{7}$$

and obtain the zonal speed u as a function of K, A, and R:

$$u = \frac{2K}{R}\left(1 - e^{-\frac{R^2}{2A^2}}\right) \tag{8}$$

For the calculation of the zonal speed, we assume that the atmosphere has a mean constant zonal flow in circular patterns moving with angular speed Ω relative to the center of the rotating coordinate system. This motion should not be confused with the angular rotation of the Earth. To a first approximation we may neglect the effect of friction on the zonal speed due to the weak radial gradients of u. Also, since r = constant, $\ddot{r} = 0$. The $\dot{\phi}$ in Equation (3a) may also be written as

$$\dot{\phi} \equiv \frac{d\phi}{dt} = \frac{ds}{dt}\frac{d\phi}{ds} = \frac{u}{r}\,.$$

Hence the meridional equation of motion Equation (3a) becomes

$$-\frac{u^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + 2u\omega\sin\lambda$$

or,

$$\frac{\partial P}{\partial r} = \rho\frac{u^2}{r} \pm 2u\omega\sin\lambda\,. \tag{9}$$

The ± sign is introduced to differentiate between zonal flow toward the East (cyclonic) and flow toward the West (anticyclonic), respectively, in the Northern Hemisphere.

Noting that $\sin\lambda = \cos\left(\frac{\mathcal{R}}{R_E}\right) \simeq \cos\left(\frac{r}{R_E}\right)$ where R_E is the Earth's radius, we may integrate the above equation:

$$\int_{P_{ref}}^{P} dP = \int_{R_{ref}}^{R}\left[\rho\frac{u^2}{r} \pm 2\rho u\omega\cos\left(\frac{r}{R_E}\right)\right] dr \tag{10}$$

where P_{ref} is a chosen reference pressure and R_{ref} is the distance from the pole where P_{ref} is measured. R in the distance from the pole of the point at which the zonal speed u is to be calculated.

The hydrostatic equation may be written:

$$\int_{P_{ref}}^{P} dP = -g\int_{z_{ref}}^{z}\rho dz = -\rho g(z - z_{ref}) \tag{11}$$

where we have assumed that ρ is equal to the average density over the pressure surface. Thus Equation (10) becomes

$$z = z_{ref} - \frac{1}{g}\int_{R_{ref}}^{R}\left[\frac{u^2(R,K,A)}{r} \pm 2u\omega\cos\left(\frac{r}{R_E}\right)\right] dr\,. \tag{12}$$

where (R_{ref}, z_{ref}) defines the position of the reference point, chosen to correspond to the position of maximum latitude on the measured isobaric surface (see Fig. 3). (R, z) are the coordinates of the point at which the zonal speed u is to be determined.

From Equation (8) we recall that u is a function of R, K and A. Equation (8) may be substituted into Equation (12) which is then solved to furnish the vortex strength K and vortex radius A as functions of the altitude z and the radial distance R from the origin.

If z and R are known on an isobaric surface, then Equation (12) may be solved numerically for K and A. Using the following notations:

n = index for a pressure surface,

m = number of data points on the n^{th} pressure surface,

i = index of data point on a pressure surface,

we may rewrite Equations (8) and (12) for the n^{th} contour as:

$$u_{i,n} = \frac{2K_n}{r_{i,n}} \left[1 - e^{.5\left(\frac{R_{i,n}}{A_n}\right)^2} \right] \qquad i = 1, \ldots, m \tag{13}$$

$$z_{i,n} = z_{ref,n} - \frac{1}{g} \int_{R_{ref,n}}^{R_{i,n}} \left[\frac{u_{i,n}^2}{r} \pm 2\omega u_{i,n} \cos\left(\frac{r}{R_E}\right) \right] dr \qquad i = 1, \ldots, m. \tag{14}$$

From pressure contour maps published by the Environmental Science Services Administration (ESSA, 1969, 1970), we obtain data points $R_{ref,n}$, $z_{i,n}$, $R_{i,n}$ ($i = 1, \ldots, m$) for the n^{th} pressure contour. Typically, the pressure contours consist of isobars at 0.4, 2, 5, 10, 30, 50, and 100 millibars.

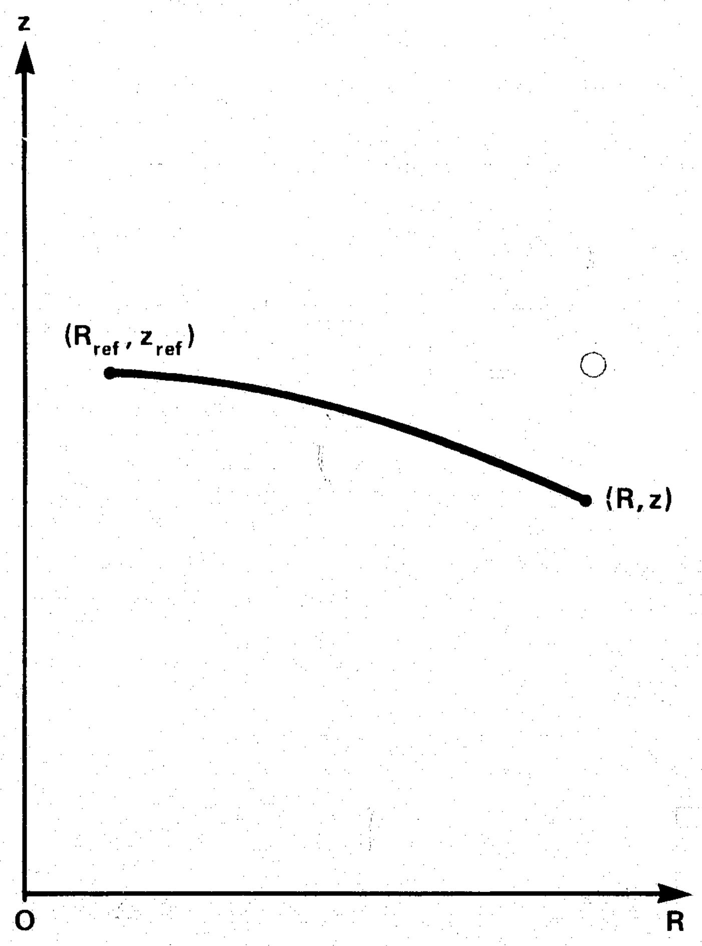

Figure 3. Illustration of Points on an Isobaric Surface Used in the Evaluation of Equation (12)

Equation (13) is used to calculate $U_{i,n}$ at each $R_{i,n}$ for the n^{th} contour. For each contour there are m values of u, which are then used as input data to Equation (14). Note that K_n and A_n are functions only of the particular contour, and do not depend on the distance from the pole. From the resulting m equations the best values of K_n and A_n can be found by the method of least squares for the n^{th} contour.

Since each contour may be considered as data taken at a certain average height, the K_n's and A_n's are therefore functions of height; curves may be fitted to them by least squares to provide K(z) and A(z) as continuous functions of height z within the altitude range of the data. From these curves the zonal wind may be calculated as a function of latitude R and height z according to Equation (8):

$$u(R,z) = \frac{2K(z)}{R}\left[1 - e^{-\frac{R^2}{2A(z)^2}}\right]. \qquad (15)$$

D. THE MERIDIONAL WIND SPEED

The effect of frictional forces on the meridional wind speed (which is roughly two orders of magnitude smaller than the zonal wind) is not negligible, as was the case for the calculation of the zonal wind. We rewrite the ϕ equation of motion Equation (3b):

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\frac{1}{\rho r}\frac{\partial P}{\partial \phi} - 2(v\omega\sin\lambda + w\omega\cos\lambda) + f\phi \qquad (3b)$$

Also, if we assume steady state motion and axial symmetry, $\dot{r} = 0$, $\ddot{\phi} = 0$ and $\frac{\partial P}{\partial \phi} = 0$. Hence the above equation becomes

$$2(v\omega\sin\lambda + w\omega\cos\lambda) = f_\phi \qquad (16)$$

The frictional forces in the atmosphere are generally thought to be similar in behavior to forces due to eddy stresses arising from turbulent motion. It can be shown that the eddy stresses give rise to a force, the ϕ component of which can be represented as (see Craig, 1960):

$$f_\phi = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\epsilon\frac{\partial u}{\partial z}\right) \tag{17}$$

where ϵ is called the coefficient of eddy viscosity.

The functional form of ϵ is not completely known. Various forms have been suggested, among which is the following due to Prandtl (Hess, 1959):

$$\epsilon = \rho l^2 \left|\frac{\partial u}{\partial z}\right| \equiv \rho\epsilon_K \tag{18}$$

where l is a characteristic mixing length, which is analogous to the molecular mean free path in a turbulence free situation, and ϵ_K is called the kinematic eddy viscosity.

Since the vertical wind speed w is small compared to the meridional wind speed v, at points near the pole the $w\omega\cos\lambda$ term in Equation (16) may be neglected, yielding:

$$2wv\sin\lambda \simeq f_\phi = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\epsilon_K\frac{\partial u}{\partial z}\right)$$

$$\text{whence} \qquad v \simeq -\frac{\frac{\partial}{\partial z}\left(\rho\epsilon_K\frac{\partial u}{\partial z}\right)}{2\rho\omega\sin\lambda} \tag{19}$$

$$= -\frac{\frac{\partial}{\partial z}\left[\rho l^2\left|\frac{\partial u}{\partial z}\right|\frac{\partial u}{\partial z}\right]}{2\rho\omega\sin\lambda}$$

Using this expression, we calculated the meridional wind v at any given location from the zonal wind shear $\frac{\partial u}{\partial z}$, since the latter was readily obtained from Equation (15) using numerical computer techniques. We note that the approximation in deriving Equation (19) improves at larger latitudes.

E. THE VERTICAL WIND SPEED

For calculating the vertical wind speed w, we make use of the equation of continuity

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} = 0 \tag{20}$$

Furthermore, if we consider the atmosphere to be in a steady state, $\frac{\partial \rho}{\partial t} = 0$, and the above equation reduces to:

$$\vec{\nabla} \cdot (\rho \vec{V}) = 0 . \tag{21}$$

In cylindrical coordinates (r, ϕ, z) the divergence of a vector $\vec{A}$ is given by:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$\therefore$ Equation (21) may be rewritten as:

$$\vec{\nabla} \cdot (\rho \vec{V}) = \frac{1}{r} \frac{\partial}{\partial r} (\rho r v) + \frac{1}{r} \frac{\partial (\rho u)}{\partial \phi} + \frac{\partial (\rho w)}{\partial z} = 0 . \tag{22}$$

From axial symmetry $\frac{\partial}{\partial \phi} (\rho u) \equiv 0$, and Equation (22) becomes

$$\frac{1}{r} \frac{\partial}{\partial r} (\rho r v) + \frac{\partial}{\partial z} (\rho w) = 0$$

or,

$$\rho \frac{v}{r} + \rho \frac{\partial v}{\partial r} + v \frac{\partial \rho}{\partial r} + w \frac{\partial \rho}{\partial z} + \rho \frac{\partial w}{\partial z} = 0 . \tag{23}$$

We assume that the equation for hydrostatic equilibrium is valid:

$$dP = - g\rho dz . \tag{24}$$

Using the ideal gas law and assuming isothermal layers (i.e., the temperature is approximately constant over the thin layer between z_1 and z_2), we may integrate the above equation to yield:

$$\ln\left(\frac{P_2}{P_1}\right) = - \frac{g}{RT} (z_2 - z_1) . \tag{25}$$

where P_1 and P_2 are pressures measured at the heights z_1 and z_2, respectively. We may therefore write:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial z} = \frac{\ln(P_2/P_1)}{z_2 - z_1} \tag{26}$$

which is substituted into Equation (23) to give

$$\frac{\partial w}{\partial z} = - \frac{v}{r} - \frac{\partial v}{\partial r} - \frac{v}{\rho} \frac{\partial \rho}{\partial r} - w \frac{\ln(P_2/P_1)}{z_2 - z_1} . \tag{27}$$

Since the density change with respect to latitude is small, the $\frac{\partial \rho}{\partial r}$ term may be neglected, and we obtain:

$$\frac{\partial w}{\partial z} = - \frac{v}{r} - \frac{\partial v}{\partial r} - w \frac{\ln(P_2/P_1)}{z_2 - z_1} . \tag{28}$$

This equation is a first order differential equation which is solved by numerical computer techniques by the method of finite differencing. To do this, the equation is expressed in finite difference form as follows:

$$\frac{w_{i+1,j} - w_{i,j}}{\Delta z} = - \frac{v_{i,j}}{r_j} - \frac{v_{i,j+1} - v_{i,j}}{\Delta r} - w_{i,j} \frac{\ln(P_{i+1}/P_i)}{z_{i+1} - z_i} \tag{29}$$

where Δz and Δr are chosen grid spacings for altitude and latitude, respectively, and i,j are level indices. The above equation is readily solved for $w_{i+1,j}$ if $w_{i,j}$ is known. Hence the boundary condition was specified to be $w_{i,j} = 0$ at an altitude of 15km to approximate the real physical situation. The values of the v's, P's and z's were known from previous calculation of the meridional speed and from the input pressure contour data.

F. INPUT DATA AND NUMERICAL COMPUTATION OF WIND SPEEDS

The algorithm for calculating the zonal, meridional and vertical wind speeds according to the polar vortex model was coded in FORTRAN and run on the IBM S/95 Computing System at Goddard Space Flight Center (GSFC). The input data for the model were extracted from pressure contour maps that exhibited nearly circular vortices around the North Pole, examples of which are given in Figures 4 and 5 for the 2mb and 0.4mb surfaces, respectively. Each chart represents a weekly or monthly average of the measured data. Figure 6 shows a summary of the input data obtained from the charts for the 0.4, 2, 5, 10, 30, 50 and 100 millibar surfaces, measured during October 1967. Each curve represents the variation of the height with latitude of each isobaric surface. It was found

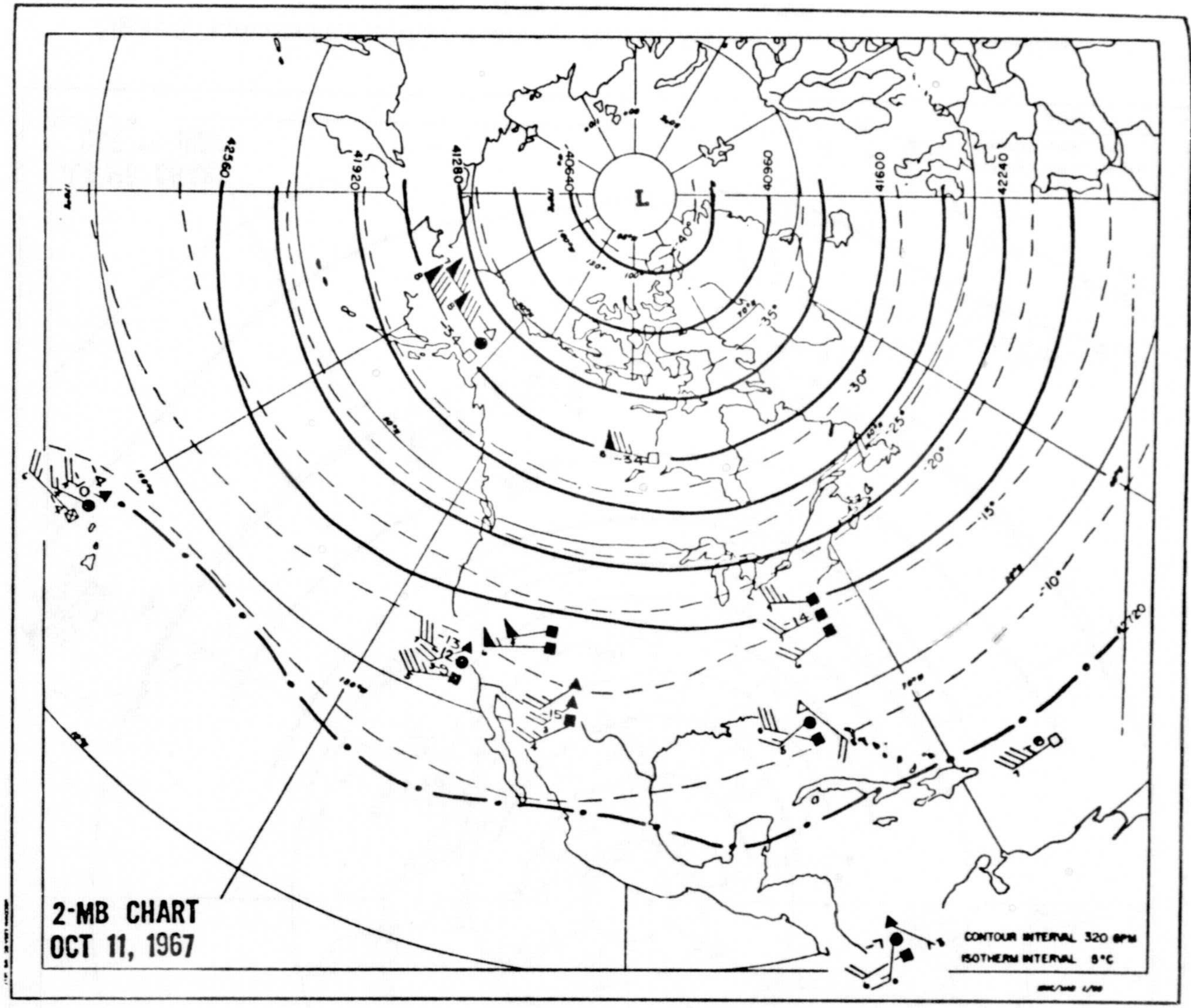

Figure 4. Contour Map of the 2-Millibar Pressure Surface, Showing Weekly Averages for Oct. 11, 1967. (ESSA Technical Report WB 11.)

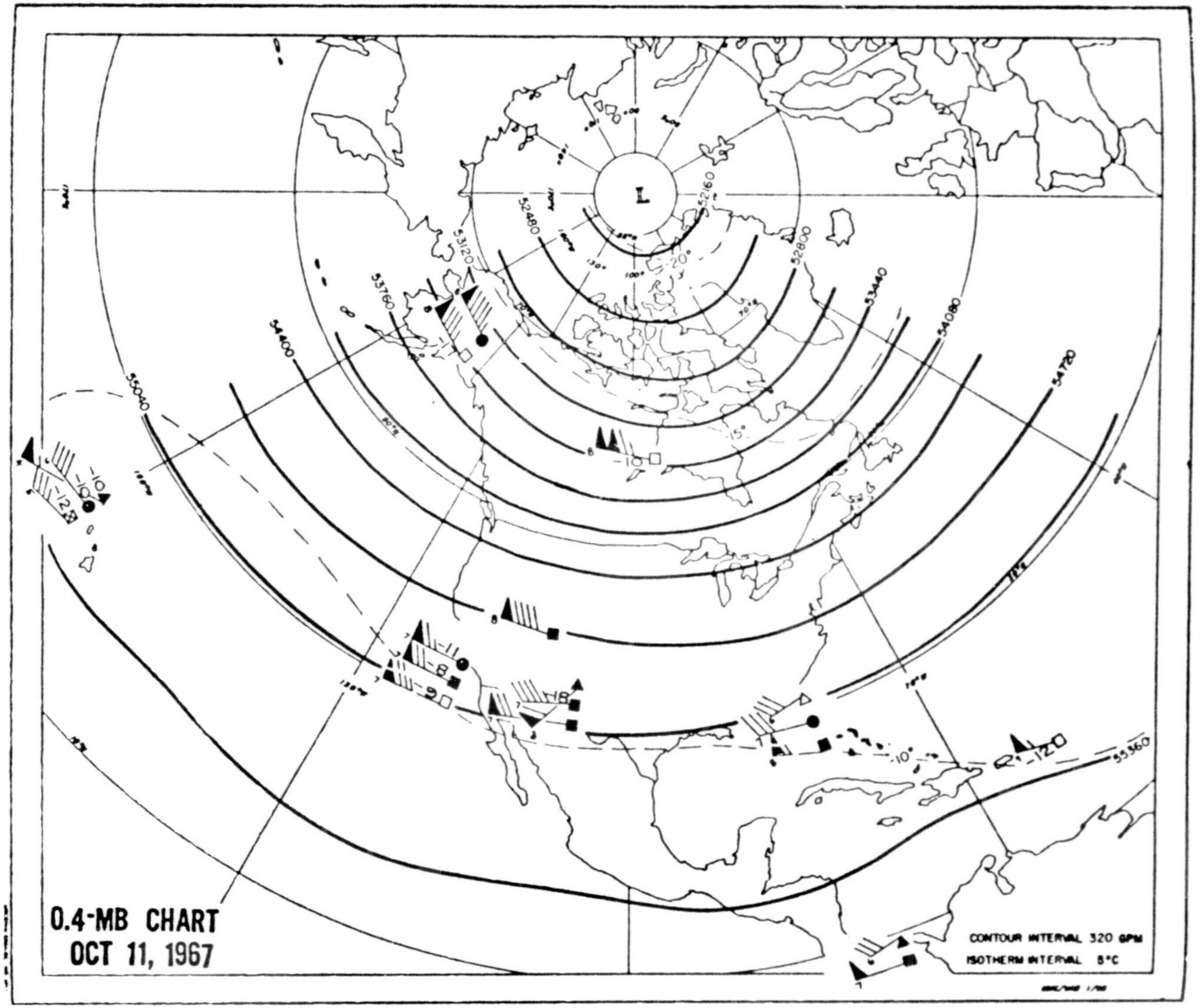

Figure 5. Contour Map of the 0.4-Millibar Pressure Surface, Showing Weekly Averages for Oct. 11, 1967. (ESSA Technical Report WB 12.)

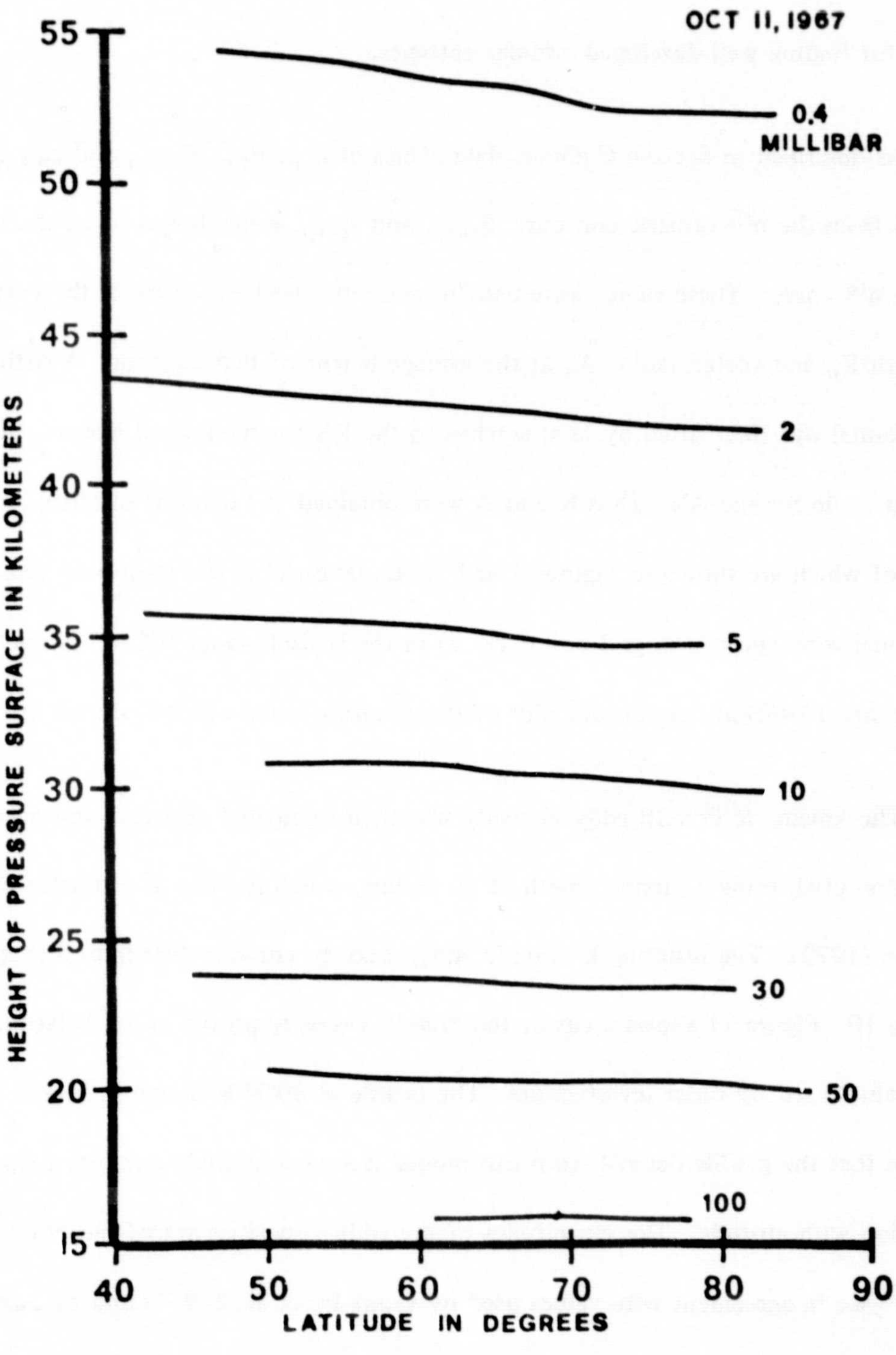

Figure 6. The Heights of Various Isobaric Surfaces as Functions of Latitude, Extracted from Contour Maps for Oct. 1967

that the late fall/winter months was the time of the year when there was the best likelihood for finding well developed circular vorticies.

As described in Section C above, data values of $z_{i,n}$, $R_{i,n}$, $R_{ref,n}$ and $z_{ref,n}$ were obtained from the n^{th} isobaric contour. $R_{ref,n}$ and $z_{ref,n}$ were chosen to be the end point of the n^{th} curve. These values were used to compute the best values of the vortex strength K_n and vortex radius A_n at the average height of that contour. A fifth degree polynomial was then fitted by least squares to the K's for the several isobars. A similar fit was made for the A's. Thus K and A were obtained as functions of altitude z, the plots of which are shown in Figures 7 and 8. Equation (15) was then used to calculate the zonal wind speed u at grid points (R, z) in the latitude range 90°–45°N and the altitude range 15–60km. A contour plot of the resulting zonal winds is shown in Figure 9.

The kinematic Prandtl eddy viscosity was then calculated at each point according to Equation (18), using a mixing length of 10 meters, which is close to estimates made by Holton (1972). The resulting kinematic eddy viscosity contour distribution is shown in Figure 10. Figure 11 shows a cut of the Prandtl viscosity profile at 40°N latitude, and also values used by other investigators. The profile at 80°N is shown in Figure 12. It is seen that the profile derived from this model at a given latitude exhibits a wavelike variation with altitude. The magnitudes of the eddy viscosities are of the order of 10 meter^2/sec in agreement with values used by Cunnold, et al., (1975) and by Justus (1973). The meridional wind speed v at each grid point is then determined from Equation (19), resulting in the contour plot shown in Figure 13.

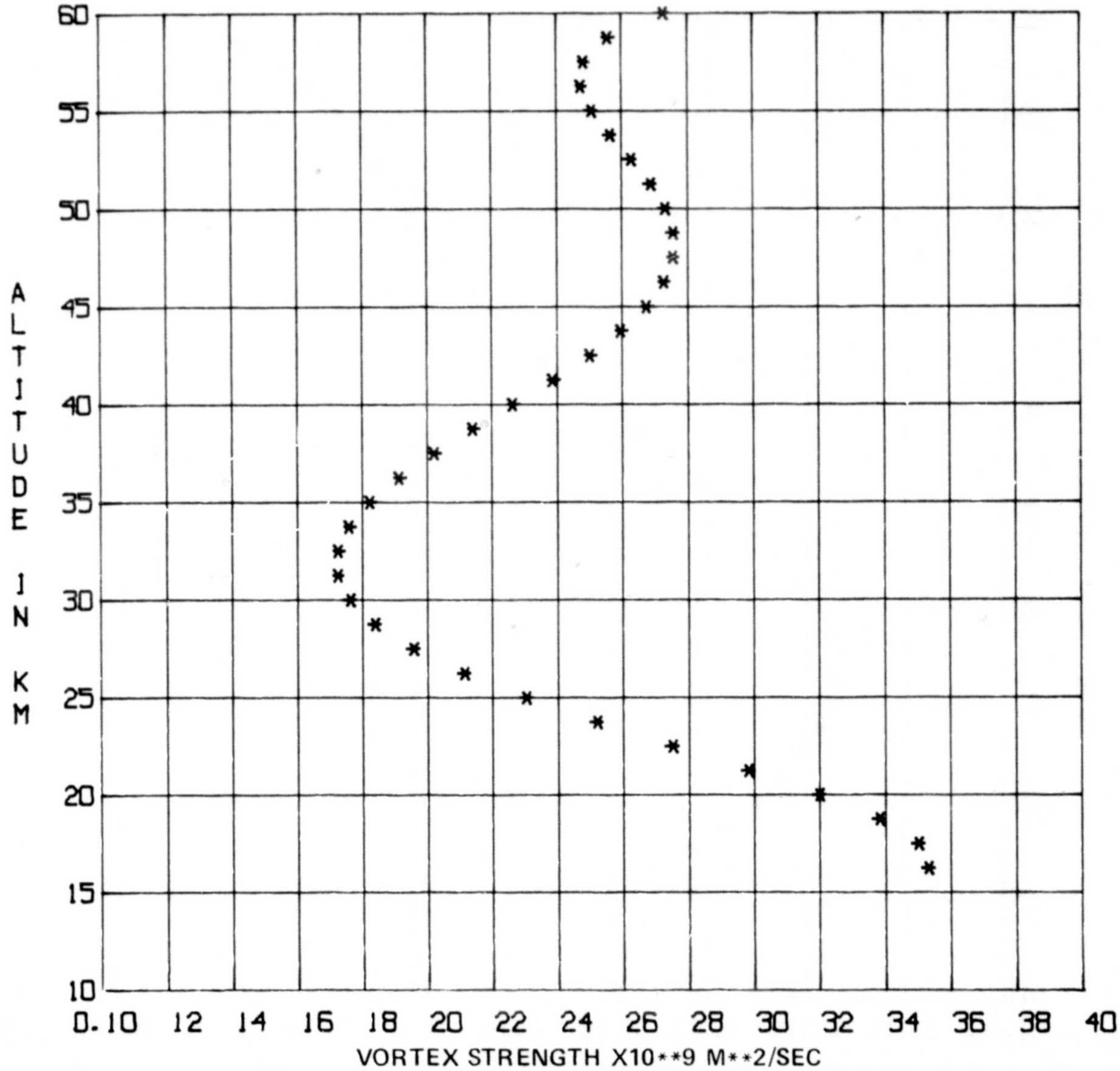

Figure 7. Vortex Strength–Altitude Profile From the Polar Vortex Wind Model

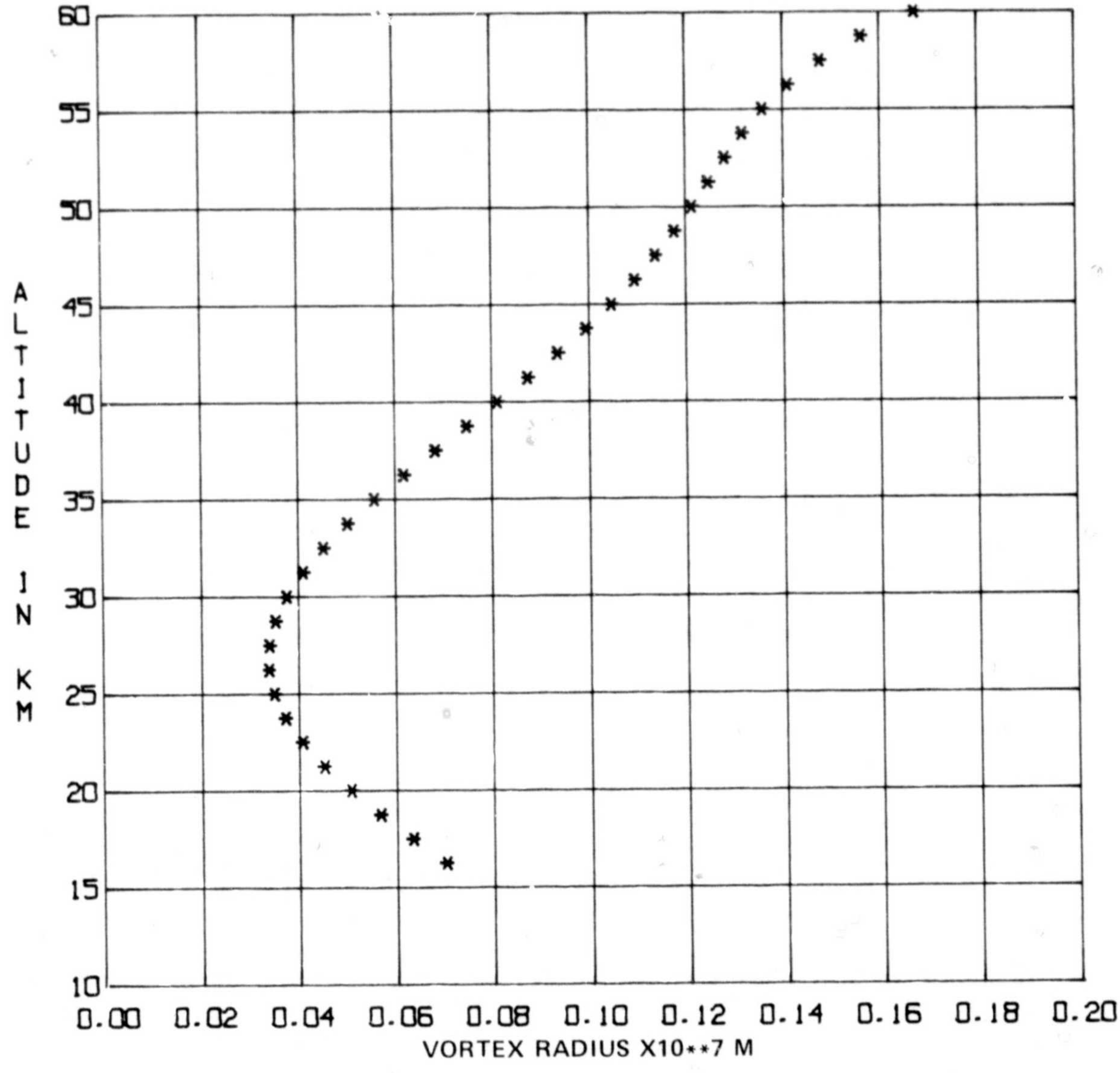

Figure 8. Vortex Radius–Altitude Profile From the Polar Vortex Wind Model

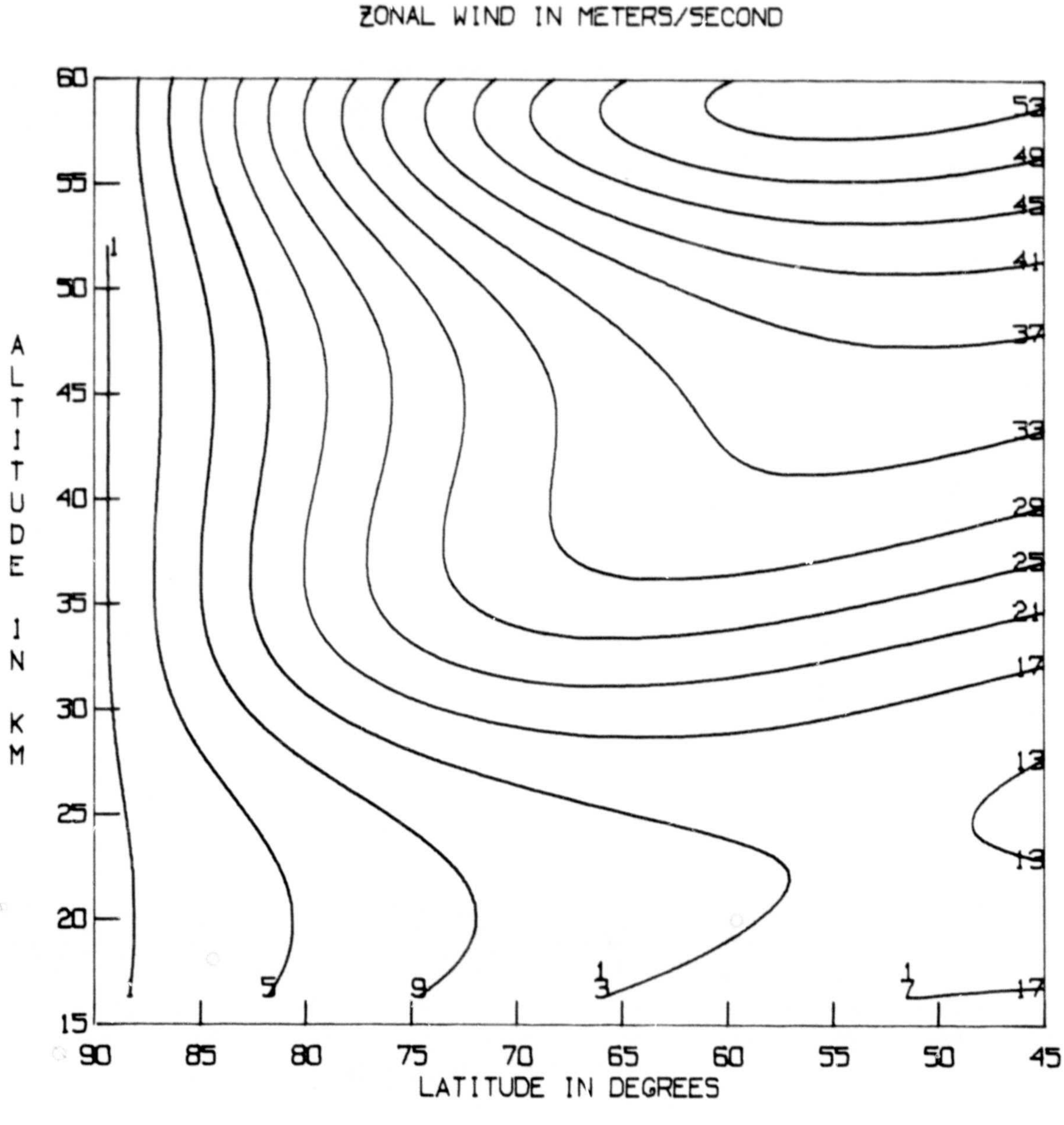

Figure 9. Contour Distribution of Zonal Wind Derived from Oct. 1967 Data

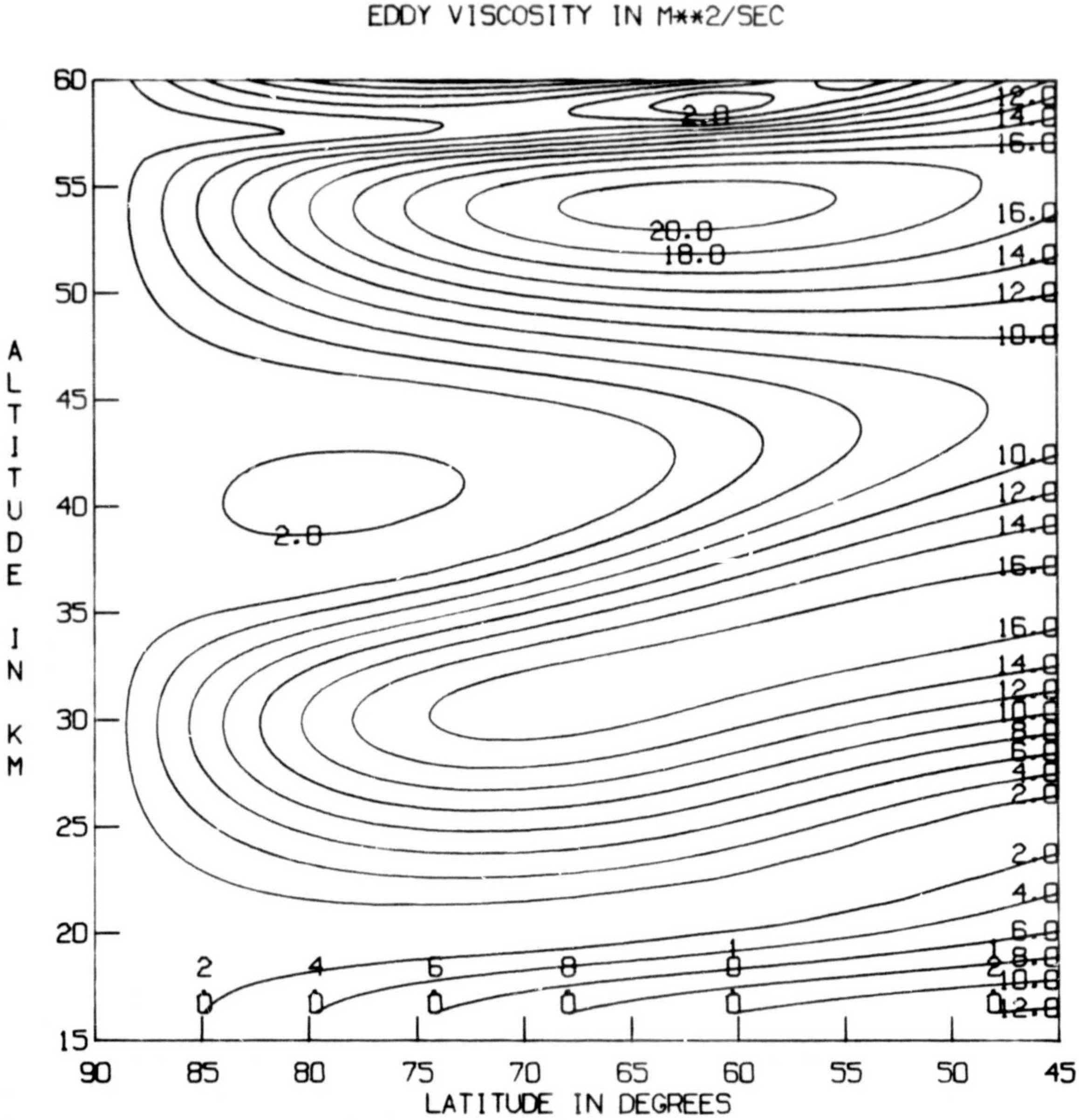

Figure 10. Contour Distribution of the Prandtl Kinematic Eddy Viscosity Derived from Oct. 1967 Data

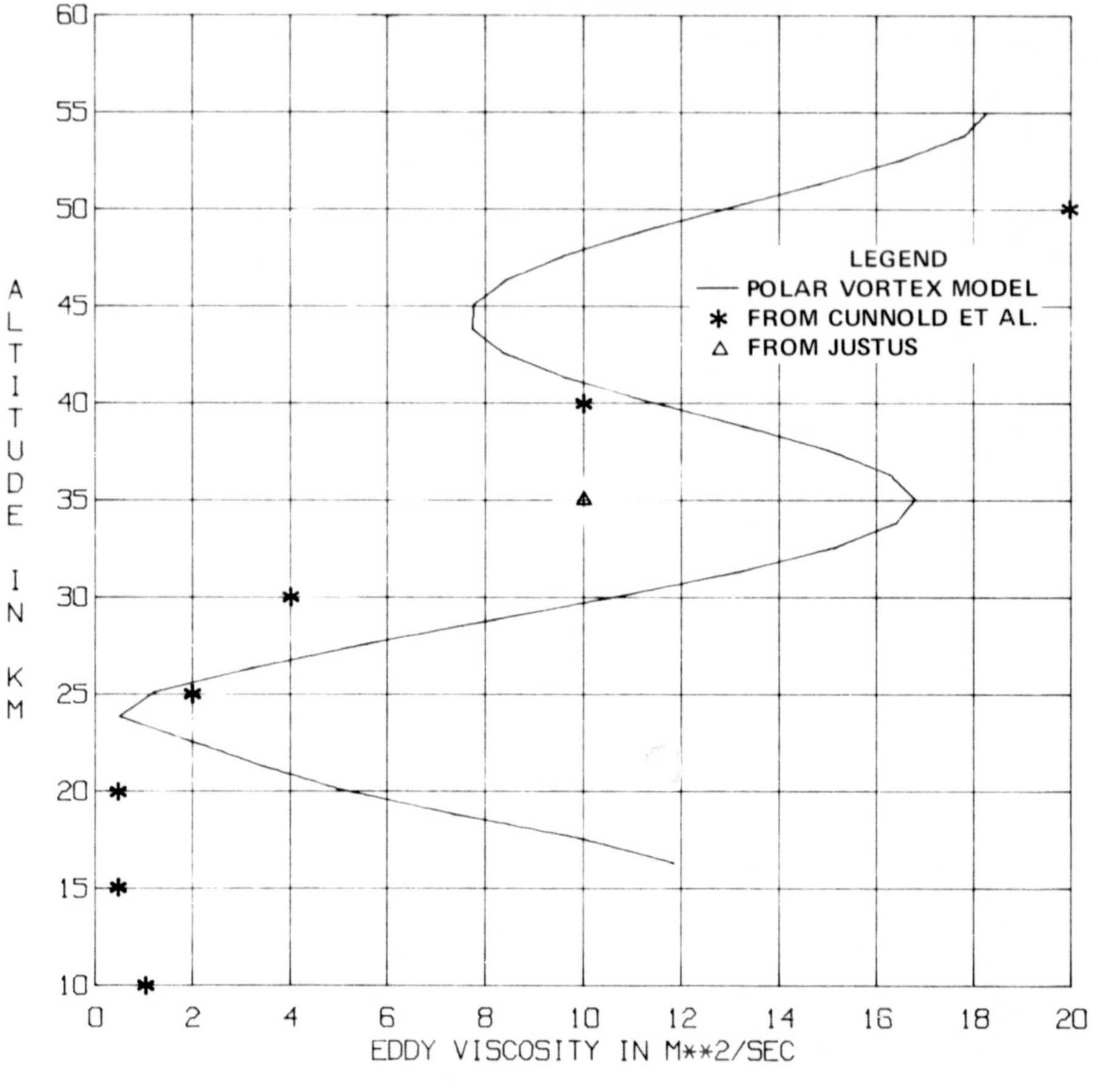

Figure 11. Prandtl Kinematic Eddy Viscosity Profile at 40°N Latitude from the Polar Vortex Wind Model Using Data from Oct. 1967. Also shown are values of eddy viscosity used by other investigators.

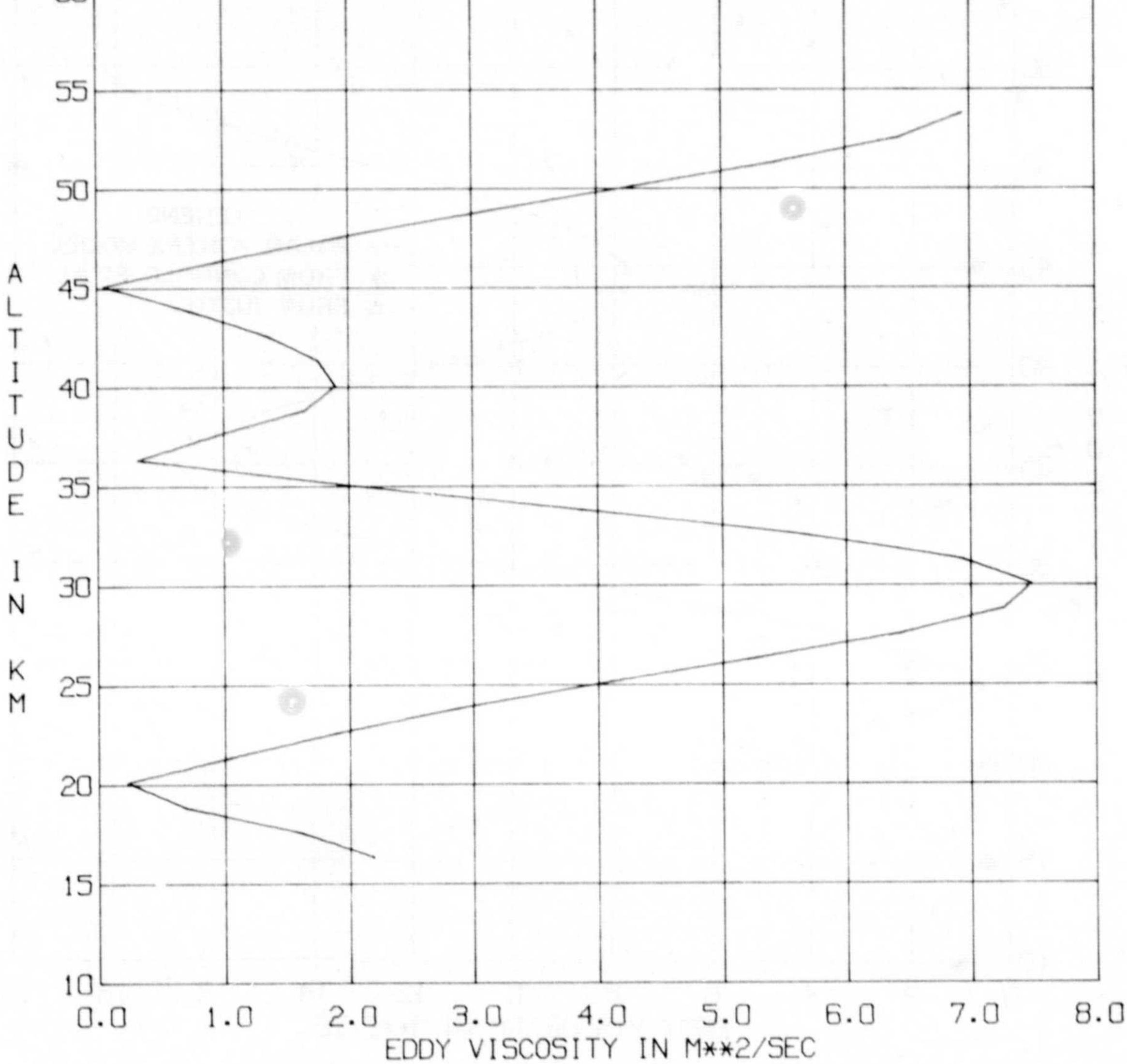

Figure 12. Prandtl Kinematic Eddy Viscosity Profile at 80°N Latitude from the Polar Vortex Wind Model Using Oct. 1967 Data

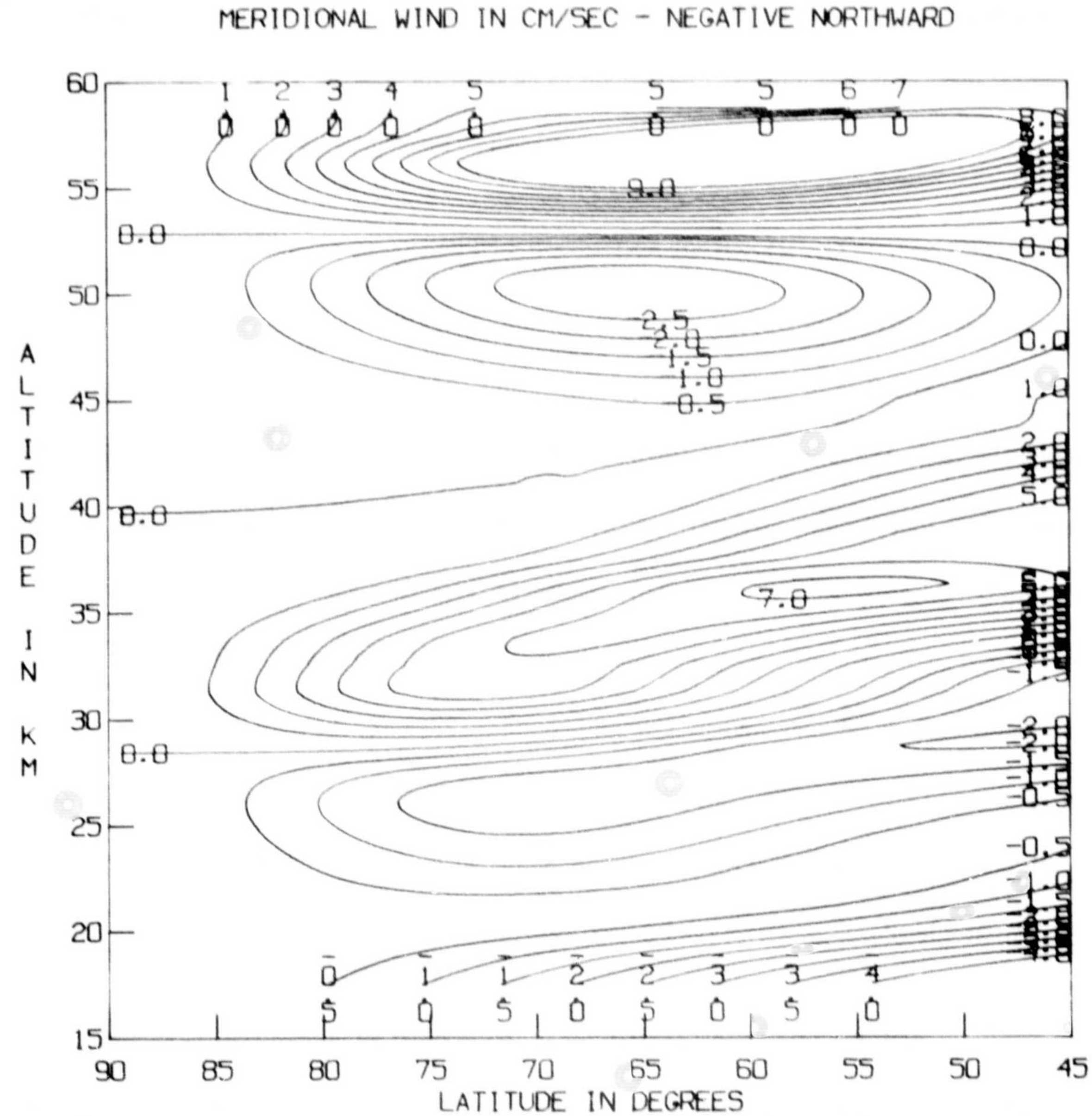

Figure 13. Meridional Wind Contour Distribution Derived from Oct. 1967 Data

The vertical winds w at the grid points were calculated by the recursion relation Equation (29), imposing the condition that the vertical wind component is zero at the lower boundary at 15km. This condition is a reasonable assumption since it is known that vertical motion is negligible at that height. The resulting vertical wind contours are given in Figure 14.

G. DISCUSSION OF RESULTS AND CONCLUSIONS

Using measured pressure contour data as input, the polar vortex wind model is used to calculate the Prandtl eddy viscosity distribution and the associated stratospheric wind components in the latitude range 90°N-45°N and at altitudes between 15 and 60km. The magnitudes of the kinematic eddy viscosity, derived by assuming an experimentally reasonable mixing length, were found to approximately agree with values used by Wofsy and McElroy (1973) in duplicating the observed distribution of methane in a one-dimensional chemical-diffusive model, and by Cunnold, et al., (1975) and Justus (1973). However, unlike the others, the eddy viscosity distribution obtained and used by the present model exhibits pronounced latitude-altitude dependence with a wavelike vertical structure in any given latitude plane.

The magnitudes of the zonal, meridional and vertical winds derived by this model were of the order of meters, centimeters and millimeters, respectively, in agreement with results published by the other investigators (Cunnold, et al., and Vincent, 1963). It was found that for cyclonic zonal flow in the Northern Hemisphere, the flow direction in a meridional plane is radially toward the pole at lower altitudes, increasing upward at intermediate

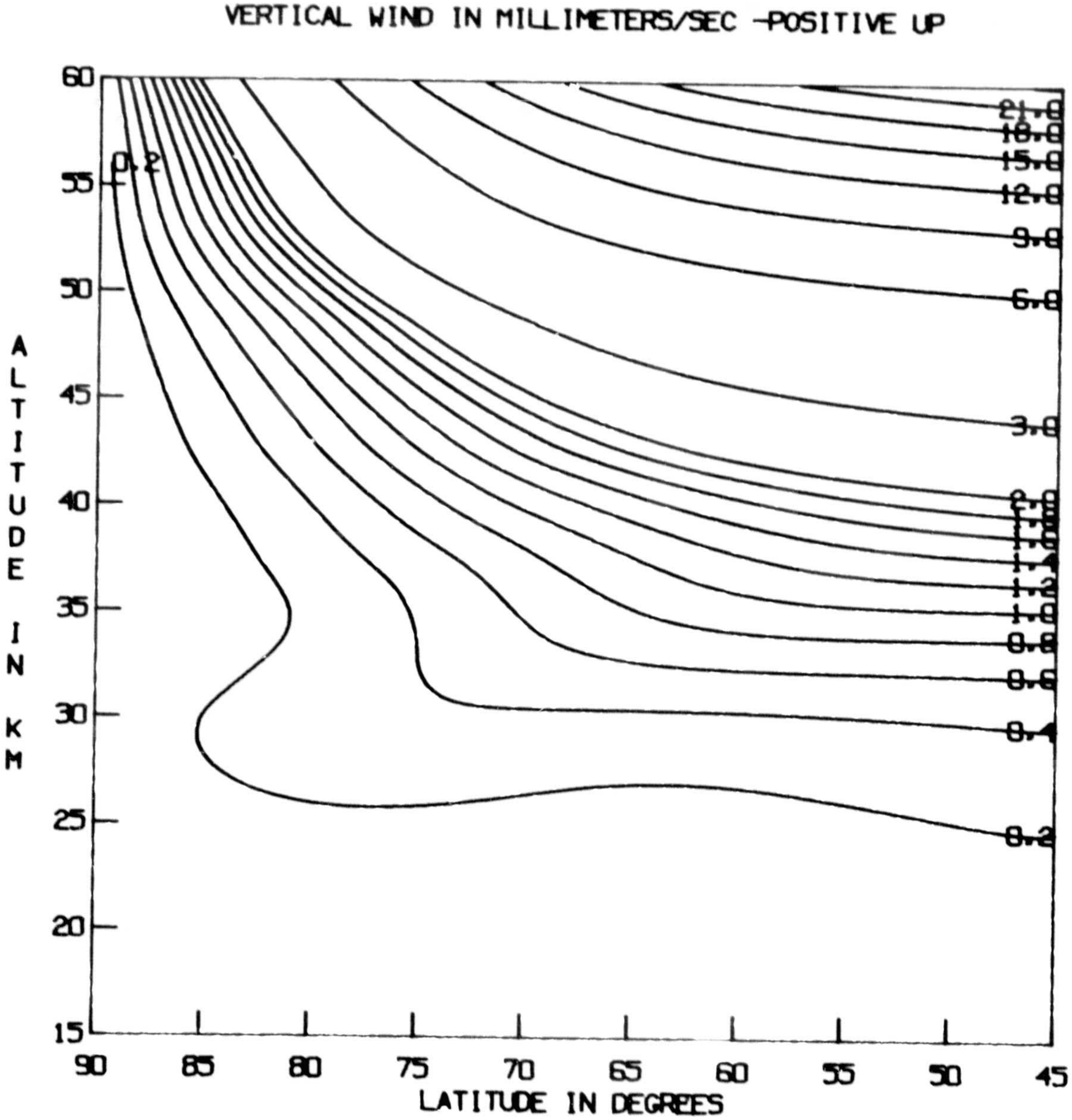

Figure 14. Vertical Wind Contour Distribution Derived from Oct. 1967 Data

levels, and radially outward at higher altitudes, as expected from fluid-dynamical considerations.

The present model is capable of utilizing experimental satellite data for deriving the corresponding eddy viscosity and wind distributions, in a semi-empirical manner. Furthermore, by being a purely fluid-dynamical model, it avoids the problem of circular reasoning characteristic of the Wofsy model. The senior author has always been ill at ease with the reasoning which derives eddy diffusivity by matching constituent profiles in photochemical models, then using the eddy diffusivity thus derived for transport in other chemical kinetic models. Also, the empirical nature of our model permits a study of variations of eddy transport with latitude and time of year, and from one year to another.

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